

Algebraic cycles on the generic abelian fourfold with polarization of type $(1, 2, 2, 2)$

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Abstract

In this paper we construct a non-trivial element in the higher Griffiths group $Griff^{3,2}$ for the generic abelian fourfold A^4 with polarization of type $(1, 2, 2, 2)$. The key idea is to use that A^4 can be realized as a generalized Prym variety and for this reason contains in a natural way some curves i.e. dimension 1 cycles.

0 Introduction

It is a difficult task to construct algebraic cycles which are homologically trivial and not algebraically equivalent to zero, i.e. non-trivial elements in the Griffiths groups

$$Griff^r := Ch_{hom}^r / Ch_{alg}^r.$$

Here Ch means always with rational coefficients, i.e. $Ch \otimes \mathbb{Q}$ and our varieties are always complex varieties. Torelli [To] has shown that abelian varieties are quotients of Jacobian varieties. This implies in particular that abelian varieties contain always 1-dimensional cycles in a natural way. Ceresa [Ce] has shown that for the generic principal polarized complex abelian variety A^3 it holds

$$0 \neq Griff^2(A^3) := Ch^2(A^3)_{hom} / Ch^2(A^3)_{alg}.$$

The word **generic** here has the following meaning (cf. [BL] pg. 559)

Definition 0.1 *A polarized abelian variety A with polarization of type (d_1, \dots, d_g) is generic for a property \mathbb{P} if $[A] \in \mathcal{A}_g(d_1, \dots, d_g)$ is outside the union of countably many proper Zariski-closed subsets of $\mathcal{A}_g(d_1, \dots, d_g)$ defined by \mathbb{P} , where $\mathcal{A}_g(d_1, \dots, d_g)$ is the moduli space of polarized abelian varieties with polarization of type (d_1, \dots, d_g) .*

Ceresa used the fact that A^3 is the Jacobian of a curve C and then applied the Abel-Jacobi map to show that the cycle $C - C^-$ is non trivial in $Griff^2(A^3)$. The explicit result is

Theorem 0.1 *Let $g \geq 3$ and $J(C)$ be generic jacobian variety of dimension g . Let W_r be the image of the natural map $Sym^r(C) \longrightarrow J(C)$ and $W_r^- := (-1_{J(C)})_* W_r$. If $1 \leq r \leq g - 2$ then the cycles $W_r - W_r^- \in Ch^{g-r}(J(C))$ are homologically but not algebraically equivalent to zero.*

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Let M be the subgroup of $\text{Griff}^2(A^3)$ generated by all the cycles of the form $C - C^- = W_1 - W_1^-$. Bardelli [Ba] and Nori [No] have shown

Theorem 0.2 *M is not finitely generated.*

We have also the following result from [Co-Pi].

Theorem 0.3 (Colombo & Pirola) $M \neq \text{Griff}^2(A^3)$.

The results presented so far use that A^3 is a jacobian variety and therefore we can't apply the methods in higher dimension. Nevertheless we can in some special cases realize abelian varieties as Prym or generalized Prym varieties which are quotients of Jacobian varieties.

This gives a motivation for looking at codimension $n - 1$ cycles on A^n , with A^n the generic (no necessary principal) polarized abelian variety of dimension n .

Fakhruddin [Fk] used the fact that the principal polarized abelian fivefold A^5 is a Prym variety corresponding to a double étale cover of a genus 9 curve over a genus 5 curve to show that

$$\text{Griff}^3(A^5) \neq 0 \neq \text{Griff}^4(A^5).$$

We write down his result

Theorem 0.4 (Fakhruddin)

- $\text{Griff}^3(A^5)$ and $\text{Griff}^4(A^5)$ are not finitely generated.
- If P is the generic Prym variety of dimension $g \geq 5$ then $\text{Griff}^j(P) \neq 0$ for $3 \leq j \leq g - 1$.
- For the generic jacobian J of dimension $g \geq 11$ we have $\text{Griff}^{g-1}(J) \neq 0$.
- In arbitrary characteristic, we have for the generic curve C of genus $g \geq 3$ that $C - C^-$ is homologically but not algebraically equivalent to zero.

The arguments in [Fk] are even strong enough to prove the result of Ceresa in arbitrary characteristic. In fact, the result of Fakhruddin is in one point better: $\text{Griff}_{(2)}^3(A^5) \neq 0 \neq \text{Griff}_{(2)}^4(A^5)$, where

$$\text{Griff}_{(i)}^r(\cdot) := \frac{\text{Ch}^r(\cdot)_{\text{hom}} \cap \text{Ch}_{(i)}^r(\cdot)}{\text{Ch}^r(\cdot)_{\text{alg}} \cap \text{Ch}_{(i)}^r(\cdot)}.$$

Here $\text{Ch}_{(i)}^r(X)$ denotes the eigenspace

$$\text{Ch}_{(i)}^r(X) := \{ \alpha \in \text{Ch}^r(X) : (m_X)_* \alpha = m^{2n-2r+i} \alpha \}.$$

We want to generalize in some sense the arguments of [Fk]. We work with the generic abelian fourfold A^4 with polarization of type $(1, 2, 2, 2)$ and consider the **higher Griffiths groups** $\text{Griff}^{r,s}$ instead of the classical ones (for the explicit definition look section at 1). Similar to Fakhruddin we consider double covers of genus 7 curves over genus 3 curves. The main difference is that we allow ramification points (Hurwitz formula implies that there are 4 ramification points). The key point is to use the fact that A^4 is a generalized Prym variety associated to such a double cover. This is a result in [BCV].

We also use ideas of [Ike] about the theory of higher infinitesimal invariants to prove the non-triviality of

$$\text{Griff}^{3,2}(A^4).$$

This paper is organized as follows: Following [Fk] we give the construction of the degeneration of A^4 . This means, we construct a family $f : X \rightarrow S$ of generalized Prym varieties such that the classifying map $S \rightarrow \mathcal{A}_4(1, 2, 2, 2)$ is dominant. We construct a relative cycle Y/S on X/S together with a subvariety $T \subset S$ in such a way that we can give an explicit description of the embedding $Y|_T \hookrightarrow X|_T$. This description will be given in the second section. Then we show that some special components of $[Y] \in H^2(S, R^4 f_* \mathbb{C})$ are non-trivial.

In the next section we show that the *second* infinitesimal invariant $\delta_2(\alpha)$ of α is non-trivial. Here α denotes the component of Y in $Ch^3_{(2)}(X/S)$ under the decomposition of Beauville (cf. [Be2] or [DeMu]) for $Ch^3(X/S)$. Using this and our results from section 4, we can show that

$$0 \neq [\alpha] \in Grif f^{3,2}(X/S).$$

We can get a refined version of this result (cf. Theorem 6.1):

Theorem 0.5 *For $s \in S$ generic we have*

$$0 \neq [\alpha_s] \in Grif f^{3,2}(A^4),$$

where A^4 is the generic abelian variety of dimension 4 with polarization of type $(1, 2, 2, 2)$.

Acknowledgments

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1 Preliminaries and notations

First we want to fix some notation. In all this section $f : X \rightarrow S$ is smooth and projective and X, S are smooth projective varieties.

The filtration of Saito and the Z -filtration

We recall the filtration F_S defined by Saito [Sa0] and its main properties.

Let $F_S^0 Ch^p(X/S) := Ch^p(X/S)$ and

$$F_S^{s+1} Ch^p(X/S) := \sum_{Y, \Gamma, q} Im \left(F_S^s Ch^{p+d-q}(Y/S) \xrightarrow{\Gamma_*} Ch^p(X/S) \right),$$

where Y, Γ and q range over all following data:

- (i) Y is projective smooth scheme, flat over S and of relative dimension d ,
- (ii) $q \in \mathbb{Z}$ satisfy $p \leq q \leq p + d$,
- (iii) $\Gamma \in Ch^q(Y \times_S X/S)$ is an algebraic cycle with the property

$$\Gamma_* \left(H_{dR}^{2(p+d-q)-s}(Y_s) \right) \subset F^{p-s+1} H_{dR}^{2p-s}(X_s);$$

where F is the Hodge filtration on the De Rham cohomology. In case $S = \text{Spec}(\mathbb{C})$ we write $F^s \text{Ch}^p(X)$ instead of $F_{\text{Spec}(\mathbb{C})}^s \text{Ch}^p(X/\text{Spec}(\mathbb{C}))$.

We state some important properties of this filtration (cf. [Sa0] and [Sa]):

Proposition 1.1 *Assume X/S is projective and smooth, then it holds:*

- SF0.** *The filtration F_S is stable under base extension and correspondences. In particular for $\alpha \in F_S^s \text{Ch}^p(X/S)$ and $s_0 \in S$ we have $\alpha_{s_0} \in F^{s_0} \text{Ch}^p(X_{s_0})$.*
- SF1.** *Inversely if $\alpha_{s_0} \in F^{s_0} \text{Ch}^p(X_{s_0})$ for $s_0 \in S$ the generic point, then there exist an open subset $U \subset S$ and an étale map $f : T \rightarrow U \subset S$ s.t. $f^*(\alpha) \in F_T^s \text{Ch}^p(T \times_S X/T)$.*
- SF2.** $F_S^{s+1} \text{Ch}^p(X/S) = F_S^s \text{Ch}^p(X/S)$ for all $s \geq p+1$.
- SF3.** $F_S^1 \text{Ch}^r(X/S) = \text{Ch}^r(X/S)_{\text{hom}} := \ker \{ \text{Ch}^r(X/S) \rightarrow H^r(X, \Omega_X^r) \}$.
- SF4.** $F_S^2 \text{Ch}^r(X/S) \subset \ker \{ AJ_X^r : \text{Ch}^r(X/S)_{\text{hom}} \rightarrow J^r(X) \}$, where $J^r(X)$ is the intermediate jacobian and AJ_X^r the Abel-Jacobi map.

The Z-Filtration

We can define an ascending filtration on the $F_S^s \text{Ch}^r(X/S)$

$$0 \subset Z_0 F_S^s \text{Ch}^r(X/S) \subset Z_1 F_S^s \text{Ch}^r(X/S) \subset \cdots \subset Z_{n-r} F_S^s \text{Ch}^r(X/S) = F_S^s \text{Ch}^r(X/S),$$

where n is the relative dimension of $f : X \rightarrow S$. This is done by

$$Z_l F_S^s \text{Ch}^r(X/S) = \sum_{Y, \Gamma} \text{Im} \{ \Gamma_* : F_S^s \text{Ch}^{d-l}(Y/S) \rightarrow \text{Ch}^r(X/S) \},$$

where Y/S ranges over all projective and smooth varieties of relative dimension d and Γ over $\text{Ch}^{l+r}(Y \times_S X/S)$.

Similar to proposition (1.1) one has:

Proposition 1.2 *Let X/S be as before.*

- Z0.** *The Z filtration on $F_S^s \text{Ch}^r(X/S)$ is stable under base extension and correspondences. In particular, for $\alpha \in Z_l F_S^s \text{Ch}^r(X/S)$ and $s_0 \in S$ we have $\alpha_{s_0} \in Z_l F^{s_0} \text{Ch}^r(X_{s_0})$.*
- Z1.** *Conversely, if $\alpha_{s_0} \in Z_l F^{s_0} \text{Ch}^r(X_{s_0})$ for $s_0 \in S$ the generic point, then there exist an open subset $U \subset S$ and an étale map $f : T \rightarrow U \subset S$ s.t. $f^*(\alpha) \in Z_l F_S^s \text{Ch}^r(T \times_S X/T)$.*
- Z2.** *For $\text{Ch}^r(X/S)_{\text{alg}}$ the subgroup of $\text{Ch}^r(X/S)$ which consist of cycles algebraically equivalent to zero holds*

$$\text{Ch}^r(X/S)_{\text{alg}} = Z_0 F_S^1 \text{Ch}^r(X/S). \quad (1)$$

The Beauville decomposition is compatible with the filtration of Saito:

Proposition 1.3 *Let X, S be smooth and connected \mathbb{C} -schemes and $f : X \rightarrow S$ smooth and projective. Then*

$$\bigoplus_{i \geq s} \text{Ch}_{(i)}^r(X/S) \subset F^s \text{Ch}^r(X/S).$$

Proof: [Mu]. □

The higher Griffiths groups

They are defined as follows

$$Griff^{r,s}(X/S) := \frac{F_S^s Ch^r(X/S)}{F_S^{s+1} Ch^r(X/S) + Z_0 F_S^s Ch^r(X/S)}.$$

From the definitions of F_S and Z we get

$$\begin{aligned} Griff^{r,1} &\simeq \left(\frac{Ch^r(X/S)_{hom}}{Ch^r(X/S)_{alg}} \right) / F_S^2 Ch^r(X/S) \\ &= \frac{Griff^r(X/S)}{F_S^2 Ch^r(X/S)}, \end{aligned}$$

where $Griff^r(X/S)$ stands for the classical Griffiths groups.

The Leray spectral sequence

It is well known that the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q f_* \mathbb{C}) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates in E_2 and in the case when f is a relative abelian scheme the induced decomposition

$$H^r(X, \mathbb{C}) \simeq \bigoplus_q H^{r-q}(S, R^q f_* \mathbb{C}). \quad (2)$$

is canonical: We can identify $H^{r-q}(S, R^q f_* \mathbb{C})$ with the subspace of $H^r(X, \mathbb{C})$ on which, for all $m \in \mathbb{Z}$, the multiplication maps $(m_{X/S})_*$ act as multiplication by m^{2n-q} . This shows

Proposition 1.4 *Let $f : X \rightarrow S$ be a relative smooth and projective abelian scheme and $cl : Ch^r(X/S) \rightarrow H^{2r}(X, \mathbb{Q}) \subset H^{2r}(X, \mathbb{C})$ be the cohomology class map. Then we have*

$$cl \left(Ch_{(s)}^r(X/S) \right) \subset H^s(S, R^{2r-s} f_* \mathbb{C}).$$

Higher infinitesimal invariants

We assume now that the fibers of $f : X \rightarrow S$ are polarized abelian varieties. We have an exact sequence

$$0 \longrightarrow f^* \Omega_S^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/S}^1 \longrightarrow 0.$$

We define the following subsheaves of Ω_X^r : $L_S^p(r) := L_S^p \Omega_X^r := f^* \Omega_S^p \wedge \Omega_X^{r-p}$ with graded pieces $Gr_L^p(r) := Gr_L^p \Omega_X^r = f^* \Omega_S^p \otimes \Omega_{X/S}^{r-p}$. It is clear that

$$\Omega_X^r = L_S^0(r) \supseteq L_S^1(r) \supseteq \cdots \supseteq L_S^r(r) \supseteq L_S^{r+1}(r) = 0.$$

With this we can give a filtration of the coherent sheaves $R^p f_* \Omega_X^r$ through

$$L^p R^l f_* \Omega_X^r := \text{Im} \left(R^l f_* L_S^p \Omega_X^r \longrightarrow R^l f_* \Omega_X^r \right).$$

We have a spectral sequence (cf. Section 5.2 of [Vo] II), the holomorphic Leray spectral sequence:

$$\mathcal{E}_1^{p,q}(r, X) := R^{p+q}f_*(Gr_L^p(r)) \Rightarrow R^{p+q}f_*\Omega_X^r \quad (3)$$

with

$$\mathcal{E}_\infty^{p,q}(r, X) = Gr_L^p R^{p+q}f_*\Omega_X^r.$$

The differential $d_1 : \mathcal{E}_1^{p,q}(r, X) \longrightarrow \mathcal{E}_1^{p+1,q}(r, X)$ is the connecting homomorphism $R^{p+q}f_*(Gr_L^p(r)) \longrightarrow R^{p+q+1}f_*(Gr_L^{p+1}(r))$ induced by

$$0 \longrightarrow Gr_L^{p+1}(r) \longrightarrow L_S^p(r)/L_S^{p+2}(r) \longrightarrow Gr_L^p(r) \longrightarrow 0.$$

Using the projection formula we can see that $\mathcal{E}_1^{p,q}(r, X) \simeq \Omega_S^p \otimes R^{p+q}f_*\Omega_{X/S}^{r-p} \simeq \Omega_S^p \otimes \mathcal{H}^{r-p, r+p}$, where $\mathcal{H}^{i,j}$ are the Hodge bundles. It can be proved (cf. [Vo] II, pg. 139) that d_1 can be identified with the map $\bar{\nabla}$ induced by ∇ , the Gauss-Manin connection at the p -th step of:

$$0 \longrightarrow Gr_F^r \mathcal{H}^{r+q} \xrightarrow{\bar{\nabla}} \Omega_S^1 \otimes Gr_F^{r-1} \mathcal{H}^{r+q} \xrightarrow{\bar{\nabla}} \Omega_S^2 \otimes Gr_F^{r-2} \mathcal{H}^{r+q} \longrightarrow \dots$$

Here F stands for the Hodge filtration. Ikeda [Ike] has shown that this spectral sequence degenerates in \mathcal{E}_2 . Now if cl denotes the composition

$$Ch^r(X/S) \xrightarrow{cl_c} H^r(X, \Omega_X^r) \longrightarrow H^0(S, R^r f_* \Omega_X^r)$$

it is possible to show (cf. [Ike] Lemma 2.6) that

$$cl(F_S^p Ch^r(X/S)) \subset H^0(S, L_S^p R^r f_* \Omega_X^r),$$

which leads to the following important definition:

Definition 1.1 For an algebraic cycle $\alpha \in F_S^s(X/S)$ we denote by $\delta_s(\alpha)$ the image of $cl(\alpha)$ under the map

$$H^0(S, L_S^s R^r f_* \Omega_X^r) \longrightarrow H^0(S, Gr_L^s R^r f_* \Omega_X^r)$$

and call it the **higher infinitesimal invariant** of α .

Moduli of double covers

Remember the definition of *stable curves*:

Definition 1.2 A genus g curve D is called a *stable curve* if the following conditions hold:

- D is connected and reduced,
- D has only ordinary double points as singularities,
- if K is a smooth rational component of D then K intersects the other components in at least 3 points.

Now we fix a natural number $n \geq 3$. Let $\overline{\mathcal{M}}_3^{(n)}$ be the moduli space of stable genus 3 curves with a level n -structure¹. We define now $\overline{\mathcal{R}}(3, 2)(n)$ through the following pullback diagram

$$\begin{array}{ccc} \overline{\mathcal{R}}(3, 2)(n) & \longrightarrow & \overline{\mathcal{M}}_3^{(n)} \\ \downarrow & & \downarrow \\ \overline{\mathcal{R}}(3, 2) & \longrightarrow & \overline{\mathcal{M}}_3 \end{array}$$

where $\mathcal{R}(3, 2)$ and $\overline{\mathcal{R}}(3, 2)$ are defined as in [BCV]. In particular we see that $\overline{\mathcal{R}}(3, 2)(n) := \overline{\mathcal{M}}_3^{(n)} \times_{\overline{\mathcal{M}}_3} \overline{\mathcal{R}}(3, 2)$. $\overline{\mathcal{R}}(3, 2)(n)$ is smooth because it is étale over the manifold $\overline{\mathcal{M}}_3^{(n)}$. It is known that there exists a universal family $\Gamma_3^{(n)} \rightarrow \overline{\mathcal{M}}_3^{(n)}$. Let $\Gamma^{(n)}$ be the pullback of $\Gamma_3^{(n)}$ under the projection map $\overline{\mathcal{R}}(3, 2)(n) \rightarrow \overline{\mathcal{M}}_3^{(n)}$.

2 Construction of the family of curves and of the generalized Prym variety

In this section we want to indicate how to modify the arguments in [Fk] to get our corresponding version of the degeneration of the generic element in $\mathcal{A}_4(1, 2, 2, 2)$.

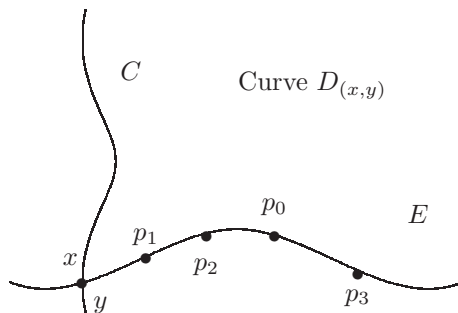
We follow the paper [Fk] for our construction of the degeneration. The main difference with Fakhruddin’s construction is that in [Fk] he considers only étale covers and here we allow ramification in 4 points. This difference is reflected in switching from $\overline{\mathcal{M}}_3^{(n)}$ to $\mathcal{R}(3, 2)(n)$.

Description of a family of curves

We fix a genus 2 curve C , an elliptic curve E and 4 points p_0, p_1, p_2, p_3 on E . We will assume that the following condition holds:

Condition: There are no Hodge classes of type $(1, 1)$ in $H^1(C, \mathbb{Q}) \otimes H^1(E, \mathbb{Q})$.

This condition is satisfied by choosing C and E generic. For each pair $(x, y) \in C \times (E - \{p_0, \dots, p_3\})$ let $D_{(x, y)}$ be the curve obtained by gluing C and E through identification of the points x and y .



¹For a definition of level n -structure we refer to [Mo].

It follows that $D_{(x,y)}$ is a genus 3 stable curve. We obtain in this way a family of stable curves $\mathcal{C} \longrightarrow C \times (E - \{p_0, \dots, p_3\})$ with a relative divisor $B := p_0 + p_1 + p_2 + p_3$. By choosing a level n -structure and a non-trivial line bundle \mathcal{L} with $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}(B)$ we get an injection

$$h : C \times (E - \{p_0, p_1, p_2, p_3\}) \longrightarrow S_1 \quad \text{with} \quad h^*(\Gamma^{(n)}) = \mathcal{C},$$

where S_1 is the open subspace of $\overline{\mathcal{R}}(3,2)(n)$ consisting of treelike curves. Let $T_1 = h(C \times (E - \{p_0, \dots, p_3\})) \subset S_1$ (so the restriction of the family to T_1 consist of curves of the form $D_{(x,y)}$) and Γ_1 the restriction of $\Gamma^{(n)} \rightarrow S_1$. Since our family consists of curves with marked points we have, in a natural way, four sections, which restricted to T_1 are given by p_0, \dots, p_3 . We will denote these sections again with p_0, \dots, p_3 .

Construction of the family of double covers

In this part we will fix one of these sections, say p_0 . We refer to the paper [Kl] of Kleiman as a reference for details of some constructions here.

For a given S_1 -morphism $g : T \longrightarrow S_1$ consider the following functors

$$\mathcal{P}ic(\Gamma_1/S_1)(T) := \left\{ \begin{array}{l} \text{Classes of isomorphisms} \\ \text{of line bundles } \mathcal{L} \text{ on} \\ \Gamma_1 \times_{S_1} T \text{ **rigidified**} \\ \text{along } p_0 \end{array} \right\}$$

and

$$\mathcal{D}iv^m(\Gamma_1/S_1)(T) := \left\{ \begin{array}{l} \text{Relative effective divisors} \\ D \text{ of } \Gamma_1 \times_{S_1} T \text{ of degree } m \end{array} \right\}.$$

The meaning of \mathcal{L} *rigidified along* p_0 is (cf. [BL] p. 597):

$$(p_0 \circ g, 1_T)^* \mathcal{L} \simeq \mathcal{O}_T.$$

The first functor is represented by a smooth algebraic space $\mathbf{Pic}_{\Gamma_1/S_1}$ of finite type over S_1 . This space is a group scheme (cf. [BLR] pg. 204).

Now we can see why it was necessary to have sections of $S_1 \longrightarrow \Gamma_1$.

Since $\Gamma_1 \longrightarrow S_1$ has curves as fibres (and is flat and projective) we have that the functor of relative effective divisors with fibres of degree m is represented by a scheme (here is necessary $m \geq 1$ too, cf. [Kl] pg. 24). Let $\mathbf{Div}_{\Gamma_1/S_1}^m$ be this scheme.

Through the 4 sections we get in a natural way a divisor B on $\Gamma_1 = \Gamma_1 \times_{S_1} S_1$ defined as

$$B := Im\{p_0 + p_1 + p_2 + p_3\} \subset \Gamma_1$$

and then a S_1 -morphism $b : S_1 \longrightarrow \mathbf{Div}_{\Gamma_1/S_1}^4$ (i.e. a section of $\mathbf{Div}_{\Gamma_1/S_1}^4 \longrightarrow S_1$). The composition

$$\mathbf{A}_{\Gamma_1/S_1}^4 \circ b \in Hom_{S_1}(S_1, \mathbf{Pic}_{\Gamma_1/S_1})$$

defines now a line bundle \mathcal{L}_0 on Γ_1 . Let ϕ be the following composition

$$\mathbf{Pic}_{\Gamma_1/S_1} \xrightarrow{2} \mathbf{Pic}_{\Gamma_1/S_1} \xrightarrow{\otimes \mathcal{L}_0^{-1}} \mathbf{Pic}_{\Gamma_1/S_1}$$

and define

$$S_2 := \ker(\phi) - (Zero) \hookrightarrow \mathbf{Pic}_{\Gamma_1/S_1}.$$

We get a S_1 -morphism

$$f : S_2 \hookrightarrow \mathbf{Pic}_{\Gamma_1/S_1} \longrightarrow S_1.$$

On $\Gamma_2 := \Gamma_1 \times_{S_1} S_2$ we then have a line bundle \mathcal{M} with the property $\mathcal{M}^{\otimes 2} \simeq \mathcal{L}_0$. We denote by the same symbol \mathcal{L}_0 also the pullback of \mathcal{L}_0 to Γ_2 by f_2 .

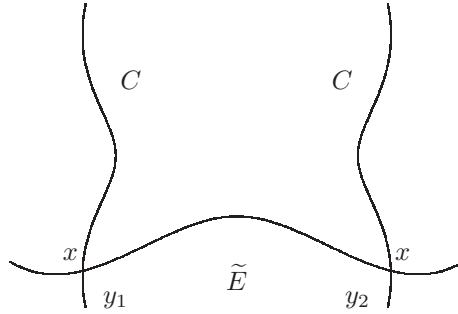
From this we get a family of double covers

$$U : \tilde{\Gamma}_2 \longrightarrow \Gamma_2$$

(with B as ramification divisor). Let T'_2 be the subset of $T_2 = f_2^{-1}(T_1)$ such that

$$\tilde{\Gamma}_2|_{T'_2} \longrightarrow T'_2$$

consists of curves of the form



where:

- \tilde{E} is a ramified double cover of E with ramification in p_0, p_1, p_2, p_3 (in particular \tilde{E} is a genus 3 curve),
- $y_1 \neq y_2$ are the elements of $U^{-1}(y)$ (in particular the 4 ramification points are in $E - \{y\}$),

Let T_3 be the connected component of T_2 which contains T'_2 . From this construction it follows in particular that $f_2|_{T_3} : T_3 \longrightarrow T_1$ is an isomorphism.

Let S_3 be the connected, Zariski-open subset of S_2 that contains T_3 and

$$\tilde{\Gamma}_3 := \tilde{\Gamma}_2|_{S_3} \longrightarrow S_3$$

consists of *treelike* curves. We use the same notation for the points in \tilde{E} determined by $p_0, \dots, p_3 \in E$ under U . Let $\tilde{T}_3 := C \times (\tilde{E} - \{p_0, \dots, p_3\})$.

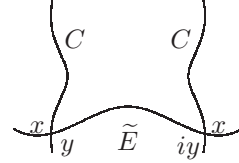
In some sense we want to *lift* the parameters from T_3 to \tilde{T}_3 . We make a base change of $\tilde{\Gamma}_3 \rightarrow S_3$ through the map $\tilde{T}_3 \xrightarrow{2:1} T_3 \hookrightarrow S_3$. Doing this we get a fibre product diagram as on the right.

$$\begin{array}{ccc} \tilde{\Gamma}_4 & \longrightarrow & \tilde{\Gamma}_3 \\ \downarrow & & \downarrow \\ \tilde{T}_4 & \longrightarrow & S_3 \end{array} \quad \text{Now}$$

we want to fix a base point of the curves to get an inclusion in the Jacobian. Therefore let D be the

divisor $\{y + iy \in (\tilde{\Gamma}_4)_{(x,y)} : y \in \tilde{E}\}$. This divisor meets a general fibre in 2 points and hence we get a generically finite surjective morphism $\tilde{T}_5 \rightarrow \tilde{T}_4$ and 2 effective divisors D_1, D_2 on $\tilde{\Gamma}_5 := \tilde{\Gamma}_4 \times_{\tilde{T}_4} \tilde{T}_5$ meeting a general fibre in only one point and such that: if $g : \tilde{\Gamma}_5 \rightarrow \tilde{\Gamma}_4$ is the natural projection then $g^*(D) = D_1 + D_2$. The geometric meaning is that we have *marked* (or choosed) one of the copies of C (we can say the left copy) by fixing D_1 . The unique points on each fibre determined by D_1 give a section of $\tilde{\Gamma}_5 \rightarrow \tilde{T}_5$. Let $f : S \rightarrow S_5$ be an étale map such that $\tilde{\Gamma} := \tilde{\Gamma}_5 \times_{S_5} S \rightarrow S$ has a section $\sigma : S \rightarrow \tilde{\Gamma}$ extending the section of $\tilde{\Gamma}_5 \rightarrow \tilde{T}_5$.

Let T be the corresponding component of $f^{-1}(\tilde{T}_5)$ such that the family $\tilde{\Gamma}|_T \rightarrow T$ consist of curves of the form as in the right picture. The families $\tilde{\Gamma} \rightarrow S$, $\tilde{\Gamma}|_T \rightarrow T$ and the section σ will be used to construct a non-trivial cycle on the generic abelian fourfold of type $(1, 2, 2, 2)$ (cf. Theorem 6.1).



2.1 Generalized Prym variety and cycle

As before $\mathbf{Pic}_{\tilde{\Gamma}/S}$ exists (since $\tilde{\Gamma} \rightarrow S$ has a section). We will follow the arguments of [Fk] pg. 113 for our construction of the (generalized) Prym variety and cycle. Let us introduce some notation first.

- $\mathbf{Pic}_{\tilde{\Gamma}/S}^0$ is the open subspace of $\mathbf{Pic}_{\tilde{\Gamma}/S}$ representing the functor of line bundles which have degree zero on each component of each fibre.
- $\mathbf{Pic}_{\tilde{\Gamma}/S}^{(j)}$, $j \in \mathbb{Z}$ is the open subspace of $\mathbf{Pic}_{\tilde{\Gamma}/S}$ corresponding to line bundles of total degree j on each fibre.
- $H \subset \mathbf{Pic}_{\tilde{\Gamma}/S}^{(0)}$ is the closure of the connected component of the identity.
- $P_c := \mathbf{Pic}_{\tilde{\Gamma}/S}^{(0)}/H$.

It is clear from the definitions that $\mathbf{Pic}_{\tilde{\Gamma}/S}^0 \subset \mathbf{Pic}_{\tilde{\Gamma}/S}^{(0)}$. We also have that

$$\phi : \mathbf{Pic}_{\tilde{\Gamma}/S}^0 \rightarrow \mathbf{Pic}_{\tilde{\Gamma}/S}^{(0)} \rightarrow P_c$$

is an isomorphism, because it is on the fibres. Let Γ_0 be the open (and dense) subspace of $\tilde{\Gamma}$ at which $\pi : \tilde{\Gamma} \rightarrow S$ is smooth. There is a natural morphism $\gamma_1 : \Gamma_0 \rightarrow \mathbf{Pic}_{\tilde{\Gamma}/S}^{(1)}$ and we can then define $\gamma_0 : \Gamma_0 \rightarrow \mathbf{Pic}_{\tilde{\Gamma}/S}^{(0)}$ by $\gamma_0(p) := \gamma_1(p) - \gamma_1(\sigma(\pi(p)))$. Let $\gamma : \tilde{\Gamma} \rightarrow \mathbf{Pic}_{\tilde{\Gamma}/S}^0$ be the following morphism

$$\Gamma_0 \xrightarrow{\gamma_0} \mathbf{Pic}_{\tilde{\Gamma}/S}^{(0)} \twoheadrightarrow P_c \xrightarrow{\phi^{-1}} \mathbf{Pic}_{\tilde{\Gamma}/S}^0$$

Because of the normality of $\tilde{\Gamma}$ we can extend γ to a morphism $\gamma : \tilde{\Gamma} \rightarrow \mathbf{Pic}_{\tilde{\Gamma}/S}^0$. The involution on $\tilde{\Gamma}$ gives another involution i on $\mathbf{Pic}_{\tilde{\Gamma}/S}^0$.

Definition 2.1 *The abelian scheme*

$$X = \text{Prym}(\tilde{\Gamma}/\Gamma) := \text{Im} \left(1 - i : \mathbf{Pic}_{\tilde{\Gamma}/S}^0 \longrightarrow \mathbf{Pic}_{\Gamma/S}^0 \right)$$

is called the **relative generalized Prym variety**.

$\pi : X \longrightarrow S$ is a (non principally) polarized abelian scheme of relative dimension 4. We have a natural cycle on X , namely

$$Y := ((1 - i) \circ \gamma) (\tilde{\Gamma}) \hookrightarrow X$$

This is a subvariety of relative codimension 3.

3 Description of the embedding $Y|_T \hookrightarrow X|_T$

This section is very important because it contains the explicit description of our cycle and this will be used to describe its cohomology class.

Observe that for $(x, y) \in T \simeq C \times (\tilde{E} - \{p_0, \dots, p_3\})$ we have $X_{(x,y)} \simeq J \times P$, where $P := \text{Prym}(\tilde{E}/E)$ is the corresponding generalized Prym variety and $J = J(C)$. Let \tilde{J} be the jacobian of \tilde{E} (\tilde{J} has dimension 3). This is an easy consequence of how $(1 - i)$ acts on each of the components of

$$\text{Pic}^0(\tilde{\Gamma}_{(x,y)}) \simeq J \times \tilde{J} \times J.$$

To describe $Y|_T \hookrightarrow X|_T$ we need first to calculate $\gamma_{(x,y)} : \tilde{\Gamma}_{(x,y)} \hookrightarrow \text{Pic}^0(\tilde{\Gamma}_{(x,y)})$ and then apply $1 - i$ to the image.

Step 1: $\gamma_{(x,y)} : \tilde{\Gamma}_{(x,y)} \hookrightarrow \text{Pic}^0(\tilde{\Gamma}_{(x,y)}) \simeq J \times \tilde{J} \times J$

Let \bar{O} be the point defined by $\sigma|_T$. We analyse now the image of a point $z \in C$. We can see that

$$\gamma_{(x,y)}(z) = [z - \bar{O}] \times 0 \times 0 = [z - x] \times 0 \times 0.$$

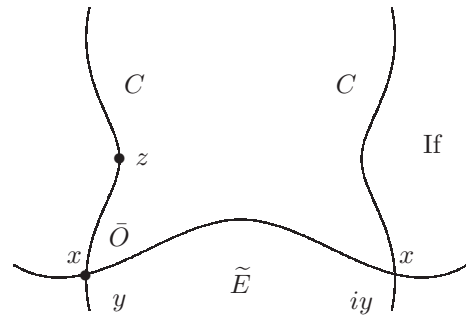
In particular

$$\gamma_{(x,y)}(y_1) = \gamma_{(x,y)}(x) = [x - x] \times [y - y] \times 0.$$

Since this description should be compatible in the points x and $y = \bar{O}$ it is necessary that the map $\tilde{E} \hookrightarrow \text{Pic}^0(\tilde{\Gamma}_{(x,y)})$ has the following form: $z \mapsto 0 \times [z - y] \times 0$.

we continue the argument in this way we get the following result:

$$\begin{aligned} C &\hookrightarrow J \times \tilde{J} \times J : & z &\mapsto [z - x] \times 0 \times 0 \\ \tilde{E} &\hookrightarrow J \times \tilde{J} \times J : & z &\mapsto 0 \times [z - y] \times 0 \\ C &\hookrightarrow J \times \tilde{J} \times J : & z &\mapsto 0 \times [iy - y] \times [z - x] \end{aligned}$$



Step 2: $(1 - i) \circ \gamma_{(x,y)} : \tilde{\Gamma}_{(x,y)} \hookrightarrow \text{Pic}^0(\tilde{\Gamma}_{(x,y)}) \longrightarrow J \times P$

We apply here the map $(1 - i)$ to the above formulas, which leads to:

- 1) $C \hookrightarrow J \times P$ is given by $z \mapsto [z - x] \times 0$,
- 2) $\tilde{E} \hookrightarrow J \times P$ is given by $z \mapsto 0 \times [y - z]$,
- 3) $C \hookrightarrow J \times P$ is given by $z \mapsto [x - z] \times [2iy - 2y]$.

This describes the embedding $Y_{(x,y)} \hookrightarrow X_{(x,y)}$ for each $(x, y) \in T$. We see that the above description makes sense over the points of $p_0, \dots, p_3 \in \tilde{E}$ and therefore for all points of $C \times \tilde{E}$. We can then extend the family and take the total space $X|_T$ as $(C \times \tilde{E}) \times (J \times P)$.

The embedding $Y|_T \hookrightarrow X|_T$

With the above fibrewise description we can give a description (now global) of $Y|_T \hookrightarrow X|_T$:

- 1) $(C \times \tilde{E}) \times C \hookrightarrow (C \times \tilde{E}) \times (J \times P)$ is given through

$$(x_1, y, x_2) \mapsto (x_1, y, [x_2 - x_1], 0),$$

- 2) $(C \times \tilde{E}) \times \tilde{E} \hookrightarrow (C \times \tilde{E}) \times (J \times P)$ is given through

$$(x, y, z) \mapsto (x, y, 0, [y - z]),$$

- 3) $(C \times \tilde{E}) \times C \hookrightarrow (C \times \tilde{E}) \times (J \times P)$ is given through

$$(x_1, y, x_2) \mapsto (x_1, y, [x_1 - x_2], [2iy - 2y]).$$

This description will be a key point for the calculation of the cohomology class of our cycle. We are interested in the component (under the Künneth decomposition) in $H^2(\tilde{E} \times C) \otimes H^4(J \times P)$ or more explicitly in

$$\left(H^1(\tilde{E}) \otimes H^1(C) \right) \otimes \left(H^1(J) \otimes H^3(P) \right).$$

4 Cohomology class of Y

We will see that the cohomology class of Y is non-trivial. For this it is enough to show that the cohomology class $[Y|_T]$ is non-trivial.

The cohomology class of $C \times \tilde{E} \times Y_{(x,y)}$ in $C \times \tilde{E} \times X_{(x,y)}$

The codimension of $C \times \tilde{E} \times Y_{(x,y)}$ in $C \times \tilde{E} \times X_{(x,y)}$ is 3. We will use the above description of the embedding to show that the cohomology class of $C \times \tilde{E} \times Y_{(x,y)}$ as element of $H^6((C \times \tilde{E}) \times X_{(x,y)}, \mathbb{Q})$ is non-trivial. Moreover that its component in

$$\left(H^1(C) \otimes H^1(\tilde{E}) \right) \otimes \left(H^1(J) \otimes H^3(P) \right)$$

is non-trivial. To begin with, observe that the only non-trivial contribution in $(H^1(C) \otimes H^1(\tilde{E})) \otimes (H^4(J \times P))$ comes from 3) of the description of the embedding.

The cycle in 1) is given as a product of the following 2 cycles:

- Z_1 : The (codimension 1) cycle $C \times C \hookrightarrow C \times J$, $(x_1, x_2) \mapsto (x_1, [x_2 - x_1])$.
- Z_2 : The (codimension 2) cycle $\tilde{E} \hookrightarrow \tilde{E} \times P$, $y \mapsto (y, [2iy - 2y])$.

Lemma 4.1 *One has*

- $0 \neq [Z_1] \in H^1(C) \otimes H^1(J)$ and
- $0 \neq [Z_2] \in H^1(\tilde{E}) \otimes H^3(P)$

Proof: We prove only the assertion about $[Z_2]$, the proof for $[Z_1]$ being similar. Let $[Z_2]^{1,3}$ be the component of $[Z_2]$ in $H^1(\tilde{E}) \otimes H^3(P)$. If $J^2(P)$ denotes the intermediate jacobian, then

$$J^2(P) \simeq H^{1,2}(P)/H^3(P, \mathbb{Z}) \simeq H^0(P, \Omega_P^1)^*/H_1(P, \mathbb{Z}) =: \text{Alb}(P) \simeq P$$

and from a result of Griffiths (cf. [Vo] I, pg. 294) we know that the following map is holomorphic:

$$AJ_P^{Z_2} : \tilde{E} \longrightarrow J^2(P) \simeq P$$

$$z \mapsto AJ_P^{Z_2}((Z_2)_z - (Z_2)_O) = [z - \bar{O}] \quad (\text{last using the identification } J^2(P) \simeq P).$$

Now from the universal property of the Jacobian \tilde{J} , the map $AJ_P^{Z_2}$ factors as in the right diagram. In according with a proposition in [Vo] I, pg. 291, the morphism ψ_{Z_2} of complex tori is induced by the morphism $[Z_2]^{1,3} \in H^1(\tilde{E}, \mathbb{Z}) \otimes H^3(P, \mathbb{Z})$.

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{AJ_P^{Z_2}} & J^n(P) \\ & \searrow & \nearrow \psi_{Z_2} \\ & \tilde{J} & \end{array}$$

Obviously the map $AJ_P^{Z_2}$ is non-trivial and therefore, from the commutativity of the diagram, one has that ψ_{Z_2} is non-zero and in particular $[Z_2]^{1,3} \neq 0$. \square

Let a be this non-trivial component of $[Y_T]$.

The primitive part of a in $(H^1(C) \otimes H^1(\tilde{E})) \otimes (H^1(J) \otimes H^3(P))$ is non-trivial

We introduce the following notation:

$$H(i, j) := (H^1(C) \otimes H^1(\tilde{E})) \otimes (H^i(J) \otimes H^j(P))$$

$$H(i) := (H^1(C) \otimes H^1(\tilde{E})) \otimes H^i(J \times P)$$

Let \mathcal{L} be a relative ample line bundle on X . From the canonical decomposition (2) we can define the image L of $c_1(\mathcal{L}) \in H^2(X, \mathbb{C})$ in $H^0(S, R^2 f_* \mathbb{C})$. With the above notation and the hard Lefschetz Theorem we get a commutative diagram

$$\begin{array}{ccc} H(2) & \xrightarrow[\simeq]{L^2} & H(6) \\ L \downarrow & & \uparrow L \\ H(4) & \xrightarrow[\simeq]{id} & H(4) \end{array}$$

and a decomposition

$$H(4) \simeq (L^2 \mathbb{P}_0 \oplus L \mathbb{P}_2) \oplus \mathbb{P}_4.$$

Here \mathbb{P}_j is a fiber of the local system $(R^j f_* \mathbb{C})_{prim}|_T$.

Warning: $H(4)$ is a direct summand of $H^6(X|_T, \mathbb{Q})$, thus

$$0 \neq a \in H(1, 3) \subset H^6(X|_T, \mathbb{Q}).$$

Proposition 4.1 *If a' is the primitive part of a in $H(1, 3)$ then $a' \neq 0$.*

Proof: Since $L^2 \mathbb{P}_0 \oplus L \mathbb{P}_2 \subset LH(2)$ it is enough to show that a is not contained in the image of

$$L : H(2) \longrightarrow H(4).$$

Remember that the polarization L is the product of the polarizations of J and P , this means $L = L_J + L_P$, where

$$L_J \in H^0(C \times \tilde{E}) \otimes (H^2(J) \otimes H^0(P))$$

is the (principal) polarization of J and

$$L_P \in H^0(C \times \tilde{E}) \otimes (H^0(J) \otimes H^2(P))$$

is the (non-principal, but of type $(1, 2)$) polarization of P .

From the hard Lefschetz Theorem we have a couple of isomorphisms:

$$L_J : H(1, 1) \simeq H(3, 1), \quad L_J^2 : H(0, 2) \simeq H(4, 2)$$

and from these the following maps are injective:

$$\begin{aligned} L_J : H(1, 1) &\longrightarrow H(3, 1) \\ L_J : H(0, 2) &\longrightarrow H(2, 2) \end{aligned} \tag{4}$$

In a similar way we get injective maps:

$$\begin{aligned} L_P : H(1, 1) &\longrightarrow H(1, 3) \\ L_P : H(2, 0) &\longrightarrow H(2, 2) \end{aligned}$$

Assume that $\alpha \in H^2$ satisfies $L\alpha = a$. Let $\alpha^{1,1}$ be its component in $H(1,1)$. Using (4) we see that it is necessary that $\alpha^{1,1} = 0$. Using this and the above other injections we get

$$a = L\alpha = L_J\alpha + L_P\alpha \in H(4,0) \oplus H(2,2) \oplus H(0,4)$$

but since $0 \neq a \in H(1,3)$ we get a contradiction. Thus we conclude that a is not in $LH(2)$. \square

Corollary 4.1 *If $T = C \times \tilde{E}$ and $X_0 = J \times P$ then the Hodge component of a' in*

$$H^0(T, \Omega_T^2) \otimes H^3(X, \Omega_{X_0}^1)_{\text{prim}}$$

is non-trivial.

Proof: After a consideration of the Hodge types and the above part our claim follows immediately from the following lemma. \square

Lemma 4.2 *There is no Hodge class in $H^1(C) \otimes H^1(\tilde{E})$.*

Proof: Assume $H^1(C) \otimes H^1(\tilde{E})$ has Hodge classes. Then there is a non-trivial morphism $J(C) \rightarrow J(\tilde{E})$ and thus an isogeny $J(\tilde{E}) \rightarrow J(C) \times E'$ for some elliptic curve E' . We denote also with \tilde{E} the image of \tilde{E} under the composition

$$\tilde{E} \hookrightarrow J(\tilde{E}) =: \tilde{J} \twoheadrightarrow J(C) \times E'.$$

Exactly as before we have

$$0 \neq [\tilde{E}] \in H^1(J(C)) \otimes H^1(E') \simeq H^1(C) \otimes H^1(E').$$

We will now prove that $H^1(E') = H^1(E)$ and this provides a contradiction with our condition about the Hodge classes of $H^1(C) \otimes H^1(E)$ (section 2).

Since E is generic we have an isogeny $\tilde{J} \rightarrow P \times E$ (cf. Remark (4.7) pg. 228 in [CvGTiB]) for some simple abelian surface P . From this and since \tilde{J} is isogenous to $J(C) \times E'$ we have an isogeny $J(C) \times E' \rightarrow P \times E$. From the non-triviality of

$$E' \hookrightarrow J(C) \times E' \twoheadrightarrow P \times E \twoheadrightarrow E$$

we can see that E' and E are isogenous. We conclude $H^1(E) = H^1(E')$. \square

The cohomology class of a remains nonzero when restricted to open subsets of $C \times \tilde{E}$

In order to conclude this stament we apply the following basic fact:

Proposition 4.2 *Let X and S be smooth algebraic varieties, $f : X \rightarrow S$ be a smooth and proper morphism and $a \in H^m(X, \mathbb{Q})$. Assume that there exists a subvariety $T \subset S$ with the following property:*

For all non-empty open subsets U of T , $i^(a) \neq 0$, where $i : f^{-1}(U) \hookrightarrow X$ is the inclusion map. Then the following holds for all non-empty open subsets V of S :*

$$j^*(a) \neq 0, \text{ where } j : f^{-1}(V) \hookrightarrow X \text{ is the inclusion map.}$$

Proof: [Fk] pg. 111. □

Next we show that the hypothesis of the above theorem is satisfied for our class a . To begin with, let $U \subset C \times E$ be open. We can assume that $Z := C \times E - U$ is a subvariety of codimension 1 (i.e. a prime divisor) because such sets determine a basis of the topology.

Let T denote the compact Kähler manifold $C \times E$. Let $\tau : \tilde{T} \rightarrow T$ be the blow-up of T along $\text{Sing}(Z)$. We can consider $U = T - Z \subset T - \text{Sing}(Z)$ as an open subset of \tilde{T} . We have the following Gysin sequence (cf. [Ku] pg. 82):

$$H^0(\tilde{Z}) \xrightarrow{\phi} H^2(\tilde{T}) \xrightarrow{\tilde{i}^*} H^2(U),$$

where $\tilde{Z} = \tilde{T} - U$ and $\tilde{i} : U \hookrightarrow \tilde{T}$ is the inclusion map. We know ([PS] Lemma 1.16) that ϕ is of type $(1, 1)$. From this fact and from our condition about the Hodge classes in $H^1(C) \otimes H^1(E)$ (section (2)) we get from the commutative diagram

$$\begin{array}{ccccc} H^0(\tilde{Z}) & \xrightarrow{\phi} & H^2(\tilde{T}) & \xrightarrow{\tilde{i}^*} & H^2(U) \\ & & \parallel & & \parallel \\ H^1(C) \otimes H^1(E) & \hookrightarrow & H^2(T) & \xrightarrow{i^*} & H^2(U) \end{array}$$

(here $i : U \hookrightarrow T$ is the inclusion map) that the restriction of i^* to

$$H^1(C) \otimes H^1(E) \rightarrow H^2(U)$$

is injective. For this reason the following map

$$i^* : H^1(C) \otimes H^1(E) \otimes H^4(J \times P) \rightarrow H^2(U) \otimes H^4(J \times P)$$

is injective. In particular we have

$$0 \neq i^*(a) \in H^6((C \times E) \times (J \times P)|_U).$$

Thus a satisfies the hypothesis of the Proposition 4.2. We then get

Theorem 4.1 *The primitive part of the cohomology of Y in $H^2(S, R^4 f_* \mathbb{Q})$ (i.e. in $H^2(S, \mathbb{P}_4)$) remains non-zero when restricted to all open subsets of S .*

Proof: The above consideration shows that $[Y] \in H^2(S, R^4 f_* \mathbb{Q})$ satisfies the hypothesis of the Proposition 4.2. □

5 About the higher infinitesimal invariant

First of all we fix some notation. For our map $f : X \longrightarrow S$ let f' be the restriction of f to $X_T := f^{-1}(T) \subset X$. Here $T \subset S$ is just as in our construction in section (2). We get a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\quad} & X_T \\ f \downarrow & & \downarrow f' \\ S & \xleftarrow{\quad} & T \end{array}$$

Consider the following commutative diagram

$$\begin{array}{ccccc} & & Y \times_S X & & \\ & p_1 \swarrow & \downarrow \phi & \searrow p_2 & \\ Y & & & & X \\ & \pi \searrow & & \swarrow f & \\ & & S & & \end{array}$$

where $\pi : Y \longrightarrow S$ is smooth and projective and Y is a smooth, projective variety of relative dimension d .

Let γ be a section of $R^u \phi_* \Omega_{Y \times_S X}^t$ and $\bar{\gamma}$ its image in $R^u \phi_* \Omega_{Y \times_S X/S}^t$ under the morphism $R^u \phi_* \Omega_{Y \times_S X}^t \longrightarrow R^u \phi_* \Omega_{Y \times_S X/S}^t$. With these notations we can formulate the following lemma:

Lemma 5.1 *A section γ of $R^u \phi_* \Omega_{Y \times_S X}^t$ induces a morphism of Leray spectral sequences (i.e. $d_1 \circ \bar{\gamma}_* = \bar{\gamma}_* \circ d_1$ on the left side):*

$$\begin{array}{ccc} \mathcal{E}_1^{p,q}(r, Y) & \xRightarrow{\quad} & R^{p+q} \pi_* \Omega_Y^r \\ \bar{\gamma}_* \downarrow & & \downarrow \gamma_* \\ \mathcal{E}_1^{p,q+u-d}(r+t-d, X) & \xRightarrow{\quad} & R^{p+q+u-d} f_* \Omega_{X/S}^{r+t-d} \end{array}$$

where $\bar{\gamma}_* = 1_{\Omega_S^p} \otimes (p_2)_*(p_1^*(-) \cdot \bar{\gamma})$ on the left side and $\gamma_* = (p_2)_*(p_1^*(-) \cdot \gamma)$ on the right side.

Proof: [Ike] Lemma 2.1. □

Example: A cycle $\Gamma \in Ch^r(Y \times_S X/S)$ induces through its cohomology class

$$\gamma = cl(\Gamma) \in H^0(S, R^r \phi_* \Omega_{Y \times_S X}^r)$$

a morphism of spectral sequences:

$$\bar{\gamma}_* : \mathcal{E}_1^{p,q}(\bullet, Y) \longrightarrow \mathcal{E}_1^{p,q+r-d}(\bullet + r - d, X).$$

We will use Lemma 5.1 in this way (look at the proofs of Lemmas 6.1 and 6.3).

Our polarization $L \in H^0(S, R^2 f_* \mathbb{C})$ from section 4 induces a section of $R^1 f_* \Omega_{X/S}^1$ and this induces morphisms (assuming $r + q \leq 4$)

$$u : \mathcal{E}_1^{p,q}(r, X) = \Omega_S^p \otimes R^{p+q} f_* \Omega_{X/S}^{r-p} \longrightarrow \Omega_S^p \otimes R^{p+q+1} f_* \Omega_{X/S}^{r-p+1} = \mathcal{E}_1^{p,q+1}(r+1, X).$$

From the hard Lefschetz theorem we get isomorphisms

$$\mathcal{E}_1^{p,q}(r, X) \xrightarrow[\simeq]{u^{4-r-q}} \mathcal{E}_1^{p,4-r}(4-q, X).$$

Definition 5.1 *The primitive part $\mathcal{E}_1^{p,q}(r, X)_{prim}$ of $\mathcal{E}_1^{p,q}(r, X)$ ($q+r \leq 5$) is defined as follows:*

$$\mathcal{E}_1^{p,q}(r, X)_{prim} := Ker \left\{ u^{5-r-q} : \mathcal{E}_1^{p,q}(r, X) \longrightarrow \mathcal{E}_1^{p,5-r}(5-q, X) \right\}.$$

Remark: Since $u = 1_{\Omega_S^p} \otimes L$, where L is as in section 4, we have

$$\mathcal{E}_1^{p,q}(r, X)_{prim} \simeq \Omega_S^p \otimes \left(R^{p+q} f_* \Omega_{X/S}^{r-p} \right)_{prim}. \quad (5)$$

5.1 The second infinitesimal invariant δ_2 of α

From now on α represents the component of $Y \in Ch^3(X/S)$ in $Ch_{(2)}^3(X/S)$ under the decomposition of Beauville (cf. [Be2] or [DeMu]). We want to show in this section that $\delta_2(\alpha) \neq 0$.

The non-triviality of $\delta_2(\alpha)$

We get from the above remarks on the Leray spectral sequence and our definition of α that $cl(\alpha)$ is contained in $H^2(S, R^4 f_* \mathbb{C})$ and from (4.1) we know that this class is non-trivial. This information is very useful for the proof of the following:

Lemma 5.2 $\delta_2(\alpha) \neq 0$. *Moreover*

$$0 \neq \delta_2(\alpha)|_T \in H^0(T, \mathcal{E}_1^{2,1}(3, X_T)_{prim}).$$

Proof: It is clear that the first claim follows from the second one, for this reason we prove only the second one.

The second claim needs some explanations because originally $\delta_2(\alpha)$ is an element of $H^0(S, \mathcal{E}_2^{2,1}(3, X))$ and there is no canonical way to lift it to $H^0(S, \mathcal{E}_1^{2,1}(3, X))$. Restricted to T , the Gauß-Manin connection ∇ is zero because of the triviality of the family $X_T \longrightarrow T$. Therefore and since $d_1 = \overline{\nabla}$ we get

$$\mathcal{E}_2^{2,1}(3, X_T) \simeq \mathcal{E}_1^{2,1}(3, X_T).$$

We have the following commutative diagram:

$$\begin{array}{ccccc} Ch_{(2)}^3(X/S) & \hookrightarrow & F_S^2 Ch^3(X/S) & \xrightarrow{\delta_2} & H^0(S, Gr_L^2 R^3 f_* \Omega_X^3) \\ \downarrow cl_{\mathbb{C}} & & & & \parallel \\ H^2(S, R^4 f_* \mathbb{C}) & & & & H^0(S, \mathcal{E}_2^{2,1}(3, X)) \\ \downarrow res|_T & & & & \downarrow res|_T \\ H^2(T) \otimes H^4(X_0, \mathbb{C}) & & & & H^0(T, \mathcal{E}_2^{2,1}(3, X_T)) \\ \downarrow proj & & & & \parallel d_1 = \overline{\nabla} = 0 \\ H^0(\Omega_T^2) \otimes H^3(X_0, \Omega_{X_0}^1) & \xlongequal{\simeq} & H^0(T, \Omega_T^2 \otimes R^3 f'_* \Omega_{X_T/T}^1) & \xlongequal{\simeq} & H^0(T, \mathcal{E}_1^{2,1}(3, X_T)) \end{array}$$

Let $cl_T(3, 1)$ be the composition of the maps in the left column and $\delta_2(\cdot)|_T$ the composition of the maps in the top row with those in the right column.

We know from (4) that

$$\begin{aligned} 0 \neq cl_T(3, 1)(\alpha) &\in H^0(\Omega_T^2) \otimes H^3(X_0, \Omega_{X_0}^1)_{prim} \\ &= H^0\left(T, \Omega_T^2 \otimes (R^3 f_* \Omega_{X_T/T}^1)_{prim}\right) \\ &= H^0(T, \mathcal{E}_1^{2,1}(3, X_T)_{prim}). \end{aligned}$$

Using this we get

$$0 \neq \delta_2(\alpha)|_T \in H^0(T, \mathcal{E}_1^{2,1}(3, X_T)_{prim}). \quad (6)$$

□

This property plays a very important role in the next section.

6 About the higher Griffiths group

This section is based on the ideas in [Ike] but we give some simplifications of the proofs in our case. Here we want to show that our cycle α induces a non-trivial element of

$$Griff^{3,2}(X_s) = \frac{F^2 Ch^3(X_s)}{F^3 Ch^r(X_s) + Z_0 F^2 Ch^3(X_s)}$$

where $s \in S$ is the generic point. We divide this task into a couple of lemmas. We know already (Proposition 1.3) that $\alpha \in F_S^2 Ch^3(X/S)$ and from this that $\alpha_s \in F^2 Ch^3(X_s)$.

Lemma 6.1 *If $s \in S$ is the generic point then $\alpha_s \notin F^3 Ch^3(X_s)$.*

Proof: We assume $\alpha_s \in F^3 Ch^3(X_s)$; this means (section 1) that

$$\alpha = \Gamma_* \beta$$

for some cycle $\beta \in F^2 Ch^{3-q+d}(Y)$, where $d = \dim Y$, and Y is a projective and smooth variety and $\Gamma \in Ch^q(Y \times X_s)$ has the property

$$\Gamma_* (H^{2d-2q+4}(Y)) \subset (F^2 H^4(X_s)).$$

Because $s \in S$ is generic by the Proposition 1.1 there is an open set $U \subset S$, an étale morphism $S' \rightarrow U \subset S$ with a point $s' \in S'$ over $s \in U \subset S$, a projective smooth morphism $Y_{S'} \rightarrow S'$ whose fiber over $s' \in S'$ is isomorphic to Y and $\beta_{S'} \in Ch^{3-q+d}(Y_{S'}/S')$ and $\Gamma_{S'} \in Ch^q(Y_{S'} \times_{S'} X_{S'}/S')$ (here $X_{S'} := X \times_S S'$) whose restrictions to Y are β and Γ respectively and the relation $\alpha_{S'} = (\Gamma_{S'})_* \beta_{S'}$ is satisfied, where $\alpha_{S'}$ is the pull-back of α by $X_{S'} \rightarrow S'$. By Theorem 4.1 we can assume $U = S$ and by abusing the notation we put $S' = S$.

From our assumption on Γ we know that

$$\Gamma_* : R^{3-q+d} f_*(\Omega_{Y_S/S}^{1-q+d}) \rightarrow R^3 f_*(\Omega_{X_S/S}^1)$$

is trivial. Therefore the induced map of spectral sequences

$$\mathcal{E}_1^{2,1-q+d}(3-q+d, Y_S) \xrightarrow{(1_{\Omega_S^2}) \otimes \Gamma_*} \mathcal{E}_1^{2,1}(3, X_S)$$

is trivial and the same holds for the induced map

$$\mathcal{E}_2^{2,1-q+d}(3-q+d, Y_S) \xrightarrow{[(1_{\Omega_S^2}) \otimes \Gamma_*]} \mathcal{E}_2^{2,1}(3, X_S).$$

Therefore the induced morphism

$$\gamma_* : H^0(S, Gr_L^2 R^{3-q+d} \pi_* \Omega_{Y_S}^{3-q+d}) \longrightarrow H^0(S, Gr_L^2 R^3 f_* \Omega_{X_S}^3)$$

is trivial too (here we use $\mathcal{E}_2^{2,\bullet} = \mathcal{E}_\infty^{2,\bullet} = Gr_L^2$).

From the following commutative diagram

$$\begin{array}{ccc} F_S^2 Ch^{3-q+d}(Y_S/S) & \xrightarrow{\Gamma_*} & F_S^2 Ch^3(X_S/S) \\ \delta_2 \downarrow & & \downarrow \delta_2 \\ H^0(S, Gr_L^2 R^{3-q+d} \pi_* \Omega_{Y_S}^{3-q+d}) & \xrightarrow{\gamma_* = 0} & H^0(S, Gr_L^2 R^3 f_* \Omega_{X_S}^3) \end{array}$$

and from Lemma 5.2 it follows that $0 \neq \delta_2(\alpha_S) = \gamma_* \delta_2(\beta_S) = 0$ and this is a contradiction. \square

We have assumed only that $\delta_2(\alpha) \neq 0$ but in fact $0 \neq \delta(\alpha)|_T \in H^0(T, \mathcal{E}_1^{2,1}(3, X)_{prim})$.

We want to use this condition. First we remark the following:

Lemma 6.2 *The map*

$$1_{\Omega_S^2} \otimes \overline{\nabla} : \mathcal{E}_1^{2,1}(3, X)_{prim} \longrightarrow \Omega_S^2 \otimes \left(\Omega_S^1 \otimes R^4 f_* (\Omega_{X/S}^0) \right)$$

is an isomorphism.

Proof: We will show that the morphism

$$\overline{\nabla} : \left(R^3 f_* \Omega_{X/S}^1 \right)_{prim} \longrightarrow \Omega_S^1 \otimes R^4 f_* \Omega_{X/S}^0$$

is an isomorphism. We do this fiberwise. To begin with, let $\omega \in H^1(X_s, \Omega_{X_s}^1)$ be the class of the polarization of X_s and $H^1(X_s, T_{X_s})_\omega = \{ \eta \in H^1(X_s, T_{X_s}) : \eta \wedge \omega = 0 \}$. We have the following exact sequence

$$0 \longrightarrow H^1(X_s, \Omega_{X_s}^3)_{prim} \longrightarrow H^1(X_s, \Omega_{X_s}^3) \xrightarrow{\wedge \omega} H^2(X_s, \Omega_{X_s}^4) \longrightarrow 0.$$

From this we get using Poincaré duality that

$$0 \longrightarrow H^2(X_s, \Omega_{X_s}^0) \xrightarrow{\wedge \omega} H^3(X_s, \Omega_{X_s}^1) \longrightarrow H^1(X_s, \Omega_{X_s}^3)^*_{prim} \longrightarrow 0.$$

This shows that

$$H^1(X_s, \Omega_{X_s}^3)^*_{prim} = H^3(X_s, \Omega_{X_s}^1)_{prim}. \quad (7)$$

From Lemma 2.2 of [Ce] we know that the wedge product

$$H^1(X_s, T_{X_s})_\omega \otimes H^0(X_s, \Omega_{X_s}^4) \longrightarrow H^1(X_s, \Omega_{X_s}^3)_{prim}$$

is surjective. By dualising we get, with help from (7), an inclusion

$$H^3(X_s, \Omega_{X_s}^1)_{prim} \hookrightarrow H^1(X_s, T_{X_s})_\omega^* \otimes H^4(X_s, \Omega_{X_s}^0) \simeq \Omega_{S,s}^1 \otimes H^4(X_s, \Omega_{X_s}^0),$$

where the right isomorphism is induced from the Kodaira-Spencer map. By a theorem of Griffiths (cf. [Vo] II pg. 136) that inclusion is exactly $\overline{\nabla}$. Now our claim follows from

$$\dim H^3(X_s, \Omega_{X_s}^1)_{prim} = 10 = \dim (\Omega_{S,s}^1 \otimes H^4(X_s, \Omega_{X_s}^0)).$$

□

From the Lefschetz decomposition for $R^3 f_* \Omega_{X/S}^1$ we can decompose $\mathcal{E}_1^{2,1}(3, X)$ as follows:

$$\mathcal{E}_1^{2,1}(3, X) \simeq \mathcal{E}_1^{2,1}(3, X)_{prim} \oplus \mathcal{V}.$$

With the above notation we have:

Lemma 6.3 *If $\alpha \in Z_0 F^2 Ch^3(X/S)$ then $\delta_2(\alpha) \in H^0(S, \mathcal{F})$ where*

$$\mathcal{F} := \frac{\mathcal{V} \cap \ker \left(d_1 : \mathcal{E}_1^{2,1}(3, X) \longrightarrow \mathcal{E}_1^{3,1}(3, X) \right)}{\mathcal{V} \cap \text{Im} \left(d_1 : \mathcal{E}_1^{1,1}(3, X) \longrightarrow \mathcal{E}_1^{3,1}(3, X) \right)}$$

and d_1 is the differential of the spectral sequence.

Proof: We assume the contrary, namely that (remember the definition of Z_0 in section 1)

$$\alpha = \Gamma_* \beta,$$

where $\beta \in F^2 Ch^d(Y/S)$, Y is a smooth projective variety of relative dimension d , and $\Gamma \in Ch^3(Y \times_S X/S)$.

Since $d = \dim(Y/S)$ it follows that $R^{d+1} f_*(\Omega_{Y/S}^{d-3}) = 0$ and from this and the commutative diagram

$$\begin{array}{ccc} \mathcal{E}_1^{2,d-2}(d, Y) = \Omega_S^2 \otimes R^d f_*(\Omega_{Y/S}^{d-2}) \xrightarrow{1_{\Omega_S^2} \otimes \Gamma_*} \Omega_S^2 \otimes R^3 f_*(\Omega_{X/S}^1) = \mathcal{E}_1^{2,1}(3, X) & & \\ \downarrow 1_{\Omega_S^2} \otimes \overline{\nabla} & & \downarrow 1_{\Omega_S^2} \otimes \overline{\nabla} \\ 0 = \Omega_S^2 \otimes \left(\Omega_S^1 \otimes R^{d+1} f_*(\Omega_{Y/S}^{d-3}) \right) \xrightarrow{1_{\Omega_S^2} \otimes \Gamma_*} \Omega_S^2 \otimes \left(\Omega_S^1 \otimes R^4 f_*(\Omega_{X/S}^0) \right) & & \end{array}$$

we conclude that

$$\left(1_{\Omega_S^2 \otimes \overline{\nabla}}\right) \left(1_{\Omega_S^2} \otimes \Gamma_*\right) \left(\mathcal{E}_1^{2,d-2}(d, Y)\right) = 0. \quad (8)$$

Let

$$\mathcal{E}_{prim}(\Gamma) := \left\{ \left(1_{\Omega_S^2} \otimes \Gamma_*\right) \left(\mathcal{E}_1^{2,d-2}(d, Y)\right) \right\} \cap \mathcal{E}_1^{2,1}(3, X)_{prim}.$$

From Lemma 6.2 and $\mathcal{E}_{prim}(\Gamma) \subset \mathcal{E}_1^{2,1}(3, X)_{prim}$ we conclude using (8) that

$$\mathcal{E}_{prim}(\Gamma) = 0.$$

Therefore the image of (we have taken cohomology)

$$\left[\left(1_{\Omega_S^2}\right) \otimes \overline{\nabla}\right] : \mathcal{E}_2^{2,d-2}(d, Y) \longrightarrow \mathcal{E}_2^{2,1}(3, X) \quad (9)$$

is contained in \mathcal{F} . Let γ_* be the map that (9) induces on global sections. Now our claim follows from the commutativity of the following diagram

$$\begin{array}{ccc} F^2Ch^d(Y/S) & \xrightarrow{\Gamma_*} & F^2Ch^3(X/S) \\ \delta_2 \downarrow & & \downarrow \delta_2 \\ H^0\left(S, \mathcal{E}_2^{2,d-2}(d, Y)\right) & \xrightarrow{\gamma_*} & H^0\left(S, \mathcal{E}_2^{2,1}(3, X)\right). \end{array}$$

□

Since d_1 is trivial over T it is clear from our definition of \mathcal{F} in Lemma 6.3 that $\mathcal{F}|_T = \mathcal{V}|_T$. This observation is a key point for the following

Lemma 6.4 $\alpha \notin Z_0 F^2Ch^3(X/S)$.

Proof: Assume it is false. Then we get from Lemma 6.3 that

$$\delta_2(\alpha)|_T \in H^0(T, \mathcal{F}|_T) = H^0(T, \mathcal{V}|_T)$$

but this (remember the definition of \mathcal{V}) contradicts Lemma 5.2. □

6.1 Main result

We present now the main result of the paper.

Theorem 6.1 *The element $\alpha \in Ch_{(2)}^3(X/S)$ gives a non-trivial element in*

$$Grif f^{3,2}(A^4),$$

where A^4 denotes the generic abelian fourfold with polarization of type $(1, 2, 2, 2)$.

Proof: Because the map $S \longrightarrow \mathcal{A}_4(1, 2, 2, 2)$ is dominant (cf. [BCV] Thm. 2.2) we see that A^4 can be realized as X_s with $s \in S$ the generic point. So the element we are looking for is α_s .

We see from Lemma 6.1 that we only need to check that $\alpha_s \notin Z_0 F^2 Ch^3(X_s)$.

Assume $\alpha_s \in Z_0 F^2 Ch^3(X_s)$. Again by Proposition 1.2, since $s \in S$ is generic, we get an open set $U \subset S$ and an étale map $\pi : S' \longrightarrow U \subset S$ such that $\alpha_{S'} := \pi^*(\alpha) \in Z_0 F^2 Ch^3(X_{S'}/S')$ where $X_{S'} = X \times_{S'} S$. Now we proceed exactly as in the proofs of the Lemmas 6.3 and 6.4. \square

Remark: The theorem holds for the generic point $s \in S$ but not over a Zariski open subset. However it will hold over the complement of a countable union of proper Zariski closed subsets. Sometimes such points are called very general. The reason is that the condition on the Hodge classes will fail over countably many proper closed subsets.

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